You should actually prove the results rather than just state them from class or your textbook. You must show your work to receive credit. Points may be withdrawn for answers given without substantiation.

1. Consider three stage explicit Runge-Kutta methods with Butcher's diagram

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ c_2 & a_{21} & 0 & 0 \\ c_3 & a_{31} & a_{32} & 0 \\ \hline & b_1 & b_2 & b_3 \end{array}$$

   (a) (6 points) What is the condition on the coefficients such that the method is second order?

   (b) (6 points) Find the region of absolute stability for this method.

   (c) (18 points) Among the identified second order methods, use the extra degree of freedom to maximize the region of absolute stability along the imaginary axis, and propose such an explicit Runge-Kutta method with this property.

2. Consider the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$. The Milne method is a linear multistep method defined by

$$y_n = y_{n-2} + \int_{t_{n-2}}^{t_n} P(t)dt$$

where P(t) is the unique quadratic polynomial that interpolates f at points t_{n-2}, t_{n-1}, t_n.

   (a) (10 points) Derive the formula for this method.

   (b) (10 points) Find the leading order term in the local truncation error of this method. What is the order of this method?

   (c) (10 points) Is this method 0-stable, strongly stable? Why?

3. For the Hamiltonian system

$$\begin{cases} p'(t) &= -H_q(p,q) \\ q'(t) &= H_p(p,q) \end{cases},$$

   (a) (10 points) Explicitly write out a first-order symplectic method.

   (b) (10 points) Explicitly write out a second-order symplectic method.

   (c) (10 points) Explicitly write out a fourth-order symplectic method.

   (d) (10 points) Briefly justify why the methods above are symplectic.